Algorithm to Construct a Regular Flat Model of an Elliptic Schemes

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1 Introduction

The purpose of this paper is to present an algorithm to construct a flat resolution of an elliptic scheme defined over a variety of characteristic zero.

I treated this topic in further generality in my PHD. thesis, and prove that such a flat resolution exists. The thesis also proves the algorithm for more general elliptic schemes.

The goal of this paper will be to succinctly describe what blowups are needed to construct the model.

First let us describe the elliptic schemes that we start with.

Definition 1.1 (Weierstrass Elliptic Scheme) Suppose $B$ is a smooth variety defined over a field of characteristic 0. Let $X$ be a variety defined over $B$ by Weierstrass equations. That is, for each open $U = \text{Spec}(R)$ of $B$, $X$ is defined as the projectivization of the subscheme of $\text{Spec}(R[x,y])$ cut out by an equation

$$y^2 = x^3 + a_4 x + a_6$$

for some $a_4, a_6 \in U$.

Such an $X \to B$ is called a Weierstrass Elliptic Scheme.

To say that $X$ has a flat resolution is the content of the following theorem.

Theorem 1.2 (Flat Resolution)

Let $X$ be a Weierstrass elliptic scheme over $B$. Then there exists a blowup $B' \to B$ and scheme $X' \to B'$ birational to $X$ such that $X' \to B'$ is regular proper and flat.

We obtain more than a resolution of $X$, which would be guaranteed by Hironaka’s desingularization theorem. In particular, the model we construct is flat over the base $B'$, and we can describe all of the fibers. We also note that not finite extension of the function field is required. We further mention that this result generalizes a result of Miranda, who dealt with two dimensional bases.
The proof of the theorem as well the construction break down into 9 steps. We shall not prove each step as in the thesis, but will rather focus on the construction of the flat resolution.

1. Reduce to the case where the reduced discriminant locus has normal crossings, and that there is a morphism $J : B \to P^1$ extending the $j$ invariant for smooth elliptic curves.

2. Assign a standard Kodaira reduction type to each component of the discriminant locus.

3. Define 3 open subschemes $S_1$, $S_2$, and $S_3$ of the base depending on the value of $J$.

4. For each component of the discriminant locus meeting an open set $S_i$, define an integer invariant depending on its Kodaira type.

5. Perform a series of blow ups of $B$ according to an algorithm which specifies the closed subschemes of $B$ of the blow ups in terms of the invariants defined above.

6. The series of blowups defines a morphism $B' \to B$. Define $X_1$ as the pullback of this morphism.

7. Construct a scheme $X_2 \to B'$ birational to $X_1$ by replacing all Weierstrass equations with minimal Weierstrass equations.

8. Order the components of the discriminant divisor, and for each component perform a series of blow ups of $X_2$. The subschemes of $X_2$ defining the blowups are higher dimensional analogs to those present in the proof of Tate’s algorithm.

9. Check that the resulting Scheme is regular and flat by examining the tangent spaces. This may be done explicitly in coordinates.

I will now give details for each step of the construction. Further discussion may be found in the thesis.
2 Preliminary Reductions

The discriminant defines a divisor $D$ in $B$. Blow up the base $B$, until the reduced preimage of $D$ has normal crossings. The fact that this process terminates is a consequence of the general resolution of singularities theorem in characteristic zero.

To define the morphism $J : B \rightarrow P^1$, just resolves the rational map $c_4^3/d$. In our case $c_4$ is just a multiple of the $a_4$ above.

3 Assigning Kodaira types

This assignment is standard and may summarized as follows.

Suppose $A$ is a component of the discriminant locus. Let $p \in A$ be a point that is not contained in any other component of the discriminant locus. Suppose that $A$ is defined locally by $t = 0$ in a neighborhood of $p$.

First, suppose $J(p) \neq \infty$, and let $v_t(d)$ be the highest power of $t$ dividing the discriminant $d$. Then we assign a Kodaira type just by examining the valuation of the discriminant as follows

\[
\begin{array}{c|c}
 v_t(d) \mod 12 & Type \\
 0 & I_0 \\
 2 & II \\
 3 & III \\
 4 & IV \\
 6 & I_0^* \\
 8 & IV^* \\
 9 & III^* \\
 10 & II^* \\
\end{array}
\]

(2)

If, on the other hand, $J(p) = \infty$ and $v_t(a_6) = 0 \mod 6$ then we assign type $I_n$ to $A$. Finally, if $J(p) = \infty$ and $v_t(a_6) = 3 \mod 6$ we assign type $I_n^*$ to
In the last two cases the integer \( n \) is given by 
\[ n = v(d) - 3v(a_6). \]

In this step we merely assigned types to components of the discriminant locus. The fact that a proper regular model of an elliptic scheme with the above valuation pattern has the special fiber described by the Kodaira type is an application of Tate’s algorithm.

The above summary can also be found in a chart in Silverman’s second volume. The fact that \( a_6 \) or \( d \) must have the above form is an easy calculation with \( a_4, a_6, \) and \( d. \)

4 Defining Subschemes

The next step is to define three open subschemes of the base which cover \( B. \)

Define \( S_1 \) to be the open subscheme of \( B \) with \( J \neq 1728 \) and \( J \neq \infty. \)

Define \( S_2 \) to be the open subscheme of \( B \) with \( J \neq 0 \) and \( J \neq \infty. \)

Define \( S_3 \) to be the open subscheme of \( B \) with \( J \neq 0 \) and \( J \neq 1728. \)

Notice that the only Kodaira types meeting \( S_1 \) are types \( \text{II}, \text{IV}, I_0^*, \text{IV}^*, \) and \( \text{II}^*. \) Also, the only Kodaira types meeting \( S_2 \) are types \( \text{III}, I_1^*, \text{III}^*, \) and the only Kodaira types meeting \( S_3 \) are types \( I_n, I_0^*, \) and \( I_n^*. \)

This follows from the standard computations of \( J. \)

Notice also that the Kodaira type \( I_0^* \) may meet more than one of the open subschemes \( S_1, S_2, \) and \( S_3. \)

5 Defining the Invariants

From this point on, we deal with the three open subschemes separately.

For each component of the discriminant divisor that meets an open subscheme \( S_i \) we associate an integer. This integer depends on the reduction
type. Let \( \lambda \) denote this association. Note that this invariant is relative to the particular open set \( S_i \).

For components in \( S_1 \) define the invariant as follows:

<table>
<thead>
<tr>
<th>Type</th>
<th>( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_0 )</td>
<td>0</td>
</tr>
<tr>
<td>( II )</td>
<td>1</td>
</tr>
<tr>
<td>( IV )</td>
<td>2</td>
</tr>
<tr>
<td>( IV^* )</td>
<td>3</td>
</tr>
<tr>
<td>( III^* )</td>
<td>4</td>
</tr>
</tbody>
</table>

6 Blowing up the base

Each blow up of the base will be entirely contained in one open set \( S_i \), and involves a separate argument in each of the three cases. Let us first focus on \( S_1 \) because it is the most interesting case.

We use the following notation: A \((5,5)\) blow up is a blow up at the reduced subscheme of \( B \) defined by two components \( A \) and \( B \) of the discriminant locus with both \( \lambda(A) = 5 \) and \( \lambda(B) = 5 \). This is namely the intersection of
two components of type $II^*$. Similarly, we write $(4,1,1)$ to denote a blow up at the reduced subscheme of $B$ defined by three components $A$, $B$ and $C$ of the discriminant locus with $\lambda(A) = 4$, $\lambda(B) = 1$, and $\lambda(C) = 1$. This is namely the intersection of two components of type $II$, and one of type $IV^*$.

Before using this notation to specify a long list of blow ups to perform, we state the remarkable fact that the exceptional divisor of the blow up has a type which is uniquely determined by the types of the components involved in the blow up.

**Proposition 6.1 (Lambda Calculation)**

Let $A$ and $B$ be components of the discriminant locus meeting at a point in $S_1$. Let $C$ be the exceptional divisor of the blow up defined by the intersection of $A$ and $B$. Then $C$ is also a component of the discriminant divisor, and

$$\lambda(C) = \lambda(A) + \lambda(B) \mod 6.$$  \hspace{1cm} (6)

The proof of this fact depends on the fact that the discriminant divisor has normal crossings and thus $a_6$ has a very special form. As an easy but representative exercise, blow up the base $C(s,t)$ of the elliptic scheme defined by $Y^2 = x^3 + st^2$ at the ideal $(s,t)$, and pull back the Weierstrass equation to the new base. The parameter defining the exceptional divisor will have exponent 3 in the new Weierstrass equation.

The same proposition is true if one blows up at the intersection of three or more components. This proposition allows us to calculate exactly the type of the exceptional divisor. For example the exceptional divisor of a $(5,5)$ blow up is a 4 type. Note that if there were a third component (say of type 2) meeting both 5 types at the same point, the new 4 type would meet this third 2 type in the blown up base.

It is also useful to blow up the base at intersections of three components. For example, the exceptional divisor of a $(4,1,1)$ blow up is a 0, $(I_0)$ type. By a judicious choice of blow ups, one can severely restrict the types of non zero components that meet.

**Proposition 6.2 (Limiting Collisions)**

The following algorithm 7 will reduce us to the case where at most two non
zero components of the discriminant locus meet, and if two non zero components do meet, they must be one of the following 3 pairs: (1,3), (1,4), or (2,3). In Kodaira notation, these collisions are (II, I_0), (II, II^*), or (IV, I_0).

The algorithm is as follows: Blow up at all collisions of the first type in the following list (7), then at all collisions of the second type, etc. After each blow up assign the appropriate lambda and kodaira type to the exceptional divisor as above.

\[
\begin{align*}
(5,5) & \quad (5,4) & \quad (5,3) & \quad (5,2) & \quad (5,1) \\
(4,4) & \quad (4,3) & \quad (4,2) & \quad (4,1,1) & \quad (3,3) \\
(1,1) & \quad (3,2,1) & \quad (2,2,2) & \quad (3,2,2) & \quad (3,2,1) \\
(2,2) & \quad (4,2) & \quad (2,1) \\
\end{align*}
\]  

(7)

To prove the proposition, one just keeps track of what new types are created at each stage and what types of collisions have been completely eliminated. In the case of \(S_1\), only (1,3), (1,4), and (2,3) pairs may be left.

I would like next briefly discuss the collisions involved in The \(S_2\) and \(S_3\) open subschemes. As above, define the \(\lambda\) invariant for each component of the discriminant locus. The series of blow ups is given by the following list:

\[
\begin{align*}
(3,3) & \quad (3,2) & \quad (3,1) & \quad (1,1) & \quad (2,1) \\
\end{align*}
\]  

(8)

In this case only (2,1) i.e. (III, I_0) collisions remain.

For the \(S_3\) open subscheme, there is a slight twist, and we need a proposition.

**Proposition 6.3 (Multiplicative Reduction)**

Let \(A\) and \(B\) be components of the discriminant locus of type \(I_m\) and \(I_n\) meeting at a point in \(S_3\). Let \(C\) be the exceptional divisor of the blow up defined by the intersection of \(A\) and \(B\). Then \(C\) is also a component of the discriminant divisor, and has type \(I_{m+n}\).

The algorithm for \(S_3\) is as follows: First perform (1,1) blow ups. Now no two \(\lambda = 1\) components meet.
Next then blow up at \((I_n, I_m)\) intersections when both \(n\) and \(m\) are odd.

Now in any remaining collision, there is at most one \(I_n^*\) type, and at most one \(I_m\) type with \(m\) odd. We may have however, any number of \(I_m\) types with \(m\) even.

7 Pullback

The series of blow ups of the base in the previous section define a composite morphism \(B' \to B\). Pull back the Scheme \(X\) via this morphism. In other words, define

\[
X_1 = X \times_B B'
\]  

(9)

8 Minimal Equations

We now construct a scheme \(X_2\) from the scheme \(X_1\) constructed above directly from the equations defining \(X_1\). We first remark that \(X_1\) is still a Weierstrass elliptic scheme, and we will use the Weierstrass equation defining it. We construct \(X_2\) locally, and later patch the schemes together.

Let \(U\) be an affine open set of \(B'\) on which \(X_1\) is defined by \(y^2 = x^3 + a_4x + a_6\). Let \(t = 0\) define a component of the discriminant locus and let \(k\) be the largest integer such that \(T^{2k}|a_4\) and \(T^{3k}|a_6\). Then define \(a'_4 = a_4/2k\) and \(a'_6 = a_6/3k\). Repeat this for each component of the discriminant locus in \(U\).

Now define \(X_2\) over \(U\) by the equation \(y^2 = x^3 + a'_4x + a'_6\). Because each component of the discriminant locus has a well defined type, these schemes patch together over an open cover of \(B'\) to define a scheme \(X_2 \to B'\). It is easy to see that the scheme \(X_2\) is birational to \(X_1\).

Note that this is nothing more than a generalization of the concept of passing to a minimal Weierstrass equation over a discrete valuation ring.
9 Desingularizing Total Space

We will now produce the regular model from the scheme $X_2$ constructed above. $X_2$ has the property that very few types of collision can occur, and it also has the minimality property of the previous section.

Essentially this step is just an application of Tate’s algorithm for each component of the discriminant divisor. The fact that the below ordering is sufficient to construct a regular model is a theorem treated in the thesis. The key item to check is that the resulting scheme is regular above points in the new base which belong to more than one component of the discriminant divisor.

Perform Tate’s algorithm for all components in the discriminant divisor of the first item in the list, then for all components of the second type, etc. One order which works is the opposite of the order that the types are usually listed:

$$II^*, III^*, IV^*, I_0^*, IV, III, II, I_n^*$$  \hspace{1cm} (10)

For convenience I specify here some (or all) of the blow ups in coordinates required by Tate’s algorithm.

Type $II$ requires no blow ups.

For types $III$ and $IV$ blow up at $(x, y, t)$.

For types $I_0^*$ and $IV^*$ blow up at first at $(x, y, t)$. Then in the third coordinate patch (I.e. $y't = y$, $x't = x$), blow up at $(y', t)$.

For types $III^*$ and $II^*$, blow up as in type $IV^*$, but further blow ups are needed in more than one subsequent coordinate patches.

$I_1$ requires no blow ups. For $I_2$ or $I_3$ blow up at $(x - \alpha, y, t)$ where $\alpha$ if a double root of $x^3 + a_4x + a_6$. For $I_n$ with $n \geq 4$ more blow ups are required in the third coordinate patch.

For $I_0^*$ first blow up as in $I_0^*$. Then at the double root of $x^3 + \frac{a_4}{4}x + \frac{a_6}{5}$. For $I_n^*$ with $n \geq 2$ more blow ups are required in the second coordinate patch(es).
10 Checking Regularity

To prove that the model is indeed regular one can check the dimensions of all of the cotangent spaces. I prove that the model is regular in my thesis. The fact that it is flat then follows from the dimensions of the fibers.

Finally I present a list of fibers of the flat model.

Over points of $B$ not on the discriminant divisor the fibers of the map $X''\to B$ are non-singular elliptic curves.

Over non-singular points of the discriminant divisor the fibers of the map $X''\to B$ are the reduction types on Kodaira’s list.

The only types of collisions that occur between reduction types are as described above, and over these singular points of the discriminant divisor the fibers of the map $X''\to B$ are also given by chart 11.

<table>
<thead>
<tr>
<th>Types in Collision</th>
<th>Special Fiber</th>
</tr>
</thead>
<tbody>
<tr>
<td>$II + I^*_0$</td>
<td>123</td>
</tr>
<tr>
<td>$II + IV^*$</td>
<td>12342</td>
</tr>
<tr>
<td>$IV + I^*_0$</td>
<td>1232</td>
</tr>
<tr>
<td>$III + I^*_0$</td>
<td>12321</td>
</tr>
<tr>
<td>$I_n + I_m$</td>
<td>$I_{n+m}$</td>
</tr>
<tr>
<td>$I_n + I^*_m$</td>
<td>$(n - odd)$</td>
</tr>
<tr>
<td>$I_n + I^*_m$</td>
<td>$(n - even)$</td>
</tr>
</tbody>
</table>

The special fibers appearing in the last column of chart 11 consist of various rational curves of given multiplicities intersecting transversally.

The $I^+_k$ type consists of 2 multiplicity 1 components connected to a chain of $k + 2$ multiplicity 2 components. This is similar to a type $I^*_k$, which has $k + 1$ multiplicity 2 components, but a pair of final multiplicity 1 components. Thus type $I^+_k$ looks like type $I^*_k$ with the final two components identified.

In the collisions specified by one of the last three lines in chart 11 there can be any number of $I_n$ types present in the collision, up to the dimension of the base scheme. However, there may be at most one $I_n$ type with $n$ odd. If there are multiple $I_n$ types colliding at a point, the special fiber is still given
by chart 11, with \( n \) replaced by \( \sum n_i \).